

THE ABELIAN ARITHMETIC REGULARITY LEMMA

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ABSTRACT. We introduce and prove the arithmetic regularity lemma of Green and Tao in the abelian case. This exposition may serve as an introduction to the general case.

The purpose of this note is to provide a brief, self-contained exposition and proof of the arithmetic regularity lemma of Green and Tao [GT10] in the abelian ($s = 1$) case, particularly in order to aid the reader of [EGM14], or to serve as an introduction to the general ($s > 1$) case. All the results of this paper can therefore be read out of the more general results expounded in the first two sections of [GT10], but to do so would require digesting the higher-order theory as well, which is a little more involved. In particular, while in [GT10] the authors rely on the inverse theorem for the U^{s+1} norm, we need only the inverse theorem for the U^2 norm, which is elementary both to state and to prove.

The results of this paper are also contained in [Tao12]. Compared to that treatment, we use slightly different language in a few places, and we absorb the Ratner-type theory into the statement of the regularity lemma.

The arithmetic regularity lemma states, roughly speaking, that an arbitrary function $f : [N] \rightarrow [0, 1]$ is the sum of a structured part f_{str} , a small part f_{sml} , and a Gowers-uniform part f_{unf} . Moreover we can buy higher-order uniformity of f_{unf} at the cost of more involved structure of f_{str} , but here we will only be able to afford U^2 uniformity.

We start with the inverse theorem for the U^2 norm. We define the $U^2(\mathbf{Z}/M\mathbf{Z})$ norm of a function $f : \mathbf{Z}/M\mathbf{Z} \rightarrow \mathbf{C}$ as

$$\|f\|_{U^2(\mathbf{Z}/M\mathbf{Z})} = \left(\mathbf{E}_{a, h_1, h_2 \in \mathbf{Z}/M\mathbf{Z}} f(a) \overline{f(a + h_1)} \overline{f(a + h_2)} f(a + h_1 + h_2) \right)^{\frac{1}{4}},$$

and then the $U^2([N])$ norm of a function $f : [N] \rightarrow \mathbf{C}$ as

$$\|f\|_{U^2([N])} = \frac{\|f\|_{U^2(\mathbf{Z}/M\mathbf{Z})}}{\|1_{[N]}\|_{U^2(\mathbf{Z}/M\mathbf{Z})}},$$

where $M \geq 2N$ and we define $f(x) = 0$ if $x \notin [N]$: one easily checks that this definition is independent of the choice of M . We will often abbreviate $U^2([N])$ to U^2 when no confusion can arise.

Given $f : \mathbf{Z}/M\mathbf{Z} \rightarrow \mathbf{C}$ we define the Fourier transform \hat{f} of f by

$$\hat{f}(r) = \mathbf{E}_{x \in \mathbf{Z}/M\mathbf{Z}} f(x) e_M(-rx)$$

for $r \in \mathbf{Z}/M\mathbf{Z}$, where $e_M(x) = e(x/M)$. The Fourier inversion formula then states

$$f(x) = \sum_{r \in \mathbf{Z}/M\mathbf{Z}} \hat{f}(r) e_M(rx).$$

Using these formulae one easily proves

$$\|f\|_{U^2(\mathbf{Z}/M\mathbf{Z})} = \left(\sum_{r \in \mathbf{Z}/M\mathbf{Z}} |\hat{f}(r)|^4 \right)^{\frac{1}{4}}.$$

Lemma 1 (Inverse theorem for the U^2 norm). *If $f : [N] \rightarrow [-1, 1]$ is a function such that $\|f\|_{U^2} \geq \delta$, then there exists $\theta \in \mathbf{T}$ such that*

$$|\mathbf{E}_{n \in [N]} f(n) e(-\theta n)| \gg_{\delta} 1.$$

Proof. The condition $\|f\|_{U^2([N])} \geq \delta$ implies that $\|f\|_{U^2(\mathbf{Z}/M\mathbf{Z})} \gg \delta$, where $M = 2N$ and as usual we extend f by zero to the rest of $\mathbf{Z}/M\mathbf{Z}$. We therefore have

$$\sum_{r \in \mathbf{Z}/M\mathbf{Z}} |\hat{f}(r)|^4 \gg \delta^4.$$

From Parseval's theorem and the hypothesis $|f| \leq 1$ it then follows that

$$\delta^4 \ll \sup |\hat{f}|^2 \left(\sum_{r \in \mathbf{Z}/M\mathbf{Z}} |\hat{f}(r)|^2 \right) = \sup |\hat{f}|^2 (\mathbf{E}_{x \in \mathbf{Z}/M\mathbf{Z}} |f(x)|^2) \leq \sup |\hat{f}|^2.$$

Thus $|\hat{f}(r)| \gg \delta^2$ for at least one $r \in \mathbf{Z}/M\mathbf{Z}$, so we may take $\theta = r/M$. \square

We need a slightly modified form of the above lemma in order to apply an energy increment argument, but first we need some language. Let us say that $f : [N] \rightarrow \mathbf{R}$ has *1-complexity* at most M if $f(n) = F(\theta n)$ for some $F : \mathbf{T}^d \rightarrow \mathbf{R}$ and $\theta \in \mathbf{T}^d$ such that $d, \|F\|_{\text{Lip}} \leq M$. Here we take the Euclidean metric

$$d(x, y) = \min_{z \in \mathbf{Z}^d} \|x - y - z\|_2$$

on \mathbf{T}^d , and we define the *Lipschitz norm* $\|F\|_{\text{Lip}}$ of $F : \mathbf{T}^d \rightarrow \mathbf{R}$ by

$$\|F\|_{\text{Lip}} = \sup_x |F(x)| + \sup_{x \neq y} \frac{|F(x) - F(y)|}{d(x, y)}.$$

The Fourier inversion formula shows that every $f : [N] \rightarrow \mathbf{C}$ has finite 1-complexity, but functions of bounded 1-complexity are special.

Our results from now on will be quantified by an arbitrary *growth function*, by which we mean simply an increasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$. By $\mathcal{F} \ll_X 1$ we will mean that \mathcal{F} is bounded by a function $\mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on the parameter X ; in other words $\mathcal{F} \ll_X 1$ means $\mathcal{F}(M) \ll_{X, M} 1$.

We say f is *1-measurable* with growth \mathcal{F} if for every $M > 0$ there is some function $f_{\text{str}} : [N] \rightarrow \mathbf{R}$ of 1-complexity at most $\mathcal{F}(M)$ such that

$$\|f - f_{\text{str}}\|_2 \leq \frac{1}{M},$$

where the $L^2([N])$ norm of a function $f : [N] \rightarrow \mathbf{C}$ is defined by

$$\|f\|_2 = (\mathbf{E}_{x \in [N]} |f(x)|^2)^{\frac{1}{2}}.$$

A set $E \subset [N]$ is called 1-measurable with growth \mathcal{F} if 1_E is so. Note that if f and g are 1-measurable with growth \mathcal{F} then $f + g$ and fg are 1-measurable with growth $\ll_{\mathcal{F}} 1$, so if E and F are 1-measurable with growth \mathcal{F} then $E \cup F$, $E \cap F$, $E \setminus F$, and so on, are all 1-measurable with growth $\ll_{\mathcal{F}} 1$.

Lemma 2 (U^2 inverse theorem, alternative formulation). *If $f : [N] \rightarrow [-1, 1]$ is a function such that $\|f\|_{U^2} \geq \delta$, then there is a 1-measurable set $E \subset [N]$ with growth $\ll_\delta 1$ such that*

$$|\mathbf{E}_{n \in [N]} f(n) 1_{E(n)}| \gg_\delta 1.$$

Proof. By the previous lemma there is some $\theta \in \mathbf{T}$ such that $\phi(n) = e(-\theta n)$ satisfies

$$|\mathbf{E}_{n \in [N]} f(n) \phi(n)| \gg_\delta 1.$$

Now by replacing ϕ with its the real or imaginary part, and then with its positive or negative part, we may assume that ϕ is real and nonnegative (e.g., if we take the real and then positive parts, then $\phi(n) = (\Re e(-\theta n))^+$).

For $0 \leq t \leq 1$, let

$$E_t = \{n \in [N] : \phi(n) \geq t\}.$$

Noting that

$$\phi(n) = \int_0^1 1_{E_t}(n) dt,$$

it follows that

$$\int_0^1 |\mathbf{E}_{n \in [N]} f(n) 1_{E_t}(n)| dt \gg_\delta 1,$$

and so

$$|\mathbf{E}_{n \in [N]} f(n) 1_{E_t}(n)| \gg_\delta 1$$

for all t in a set $\Omega \subset [0, 1]$ of measure $|\Omega| \gg_\delta 1$.

Among these sets E_t with $t \in \Omega$ there must be some E_t which is approximately invariant under small changes in t . Indeed, if

$$M(t) = \sup_{r>0} \frac{1}{2r} \frac{1}{N} |\{n \in [N] : |\phi(n) - t| \leq r\}|$$

then the Hardy–Littlewood maximal inequality (see any standard reference, such as [Rud87]) states

$$|\{t \in [0, 1] : M(t) \geq \lambda\}| \ll \frac{1}{\lambda}.$$

Since $|\Omega| \gg_\delta 1$ there is some $t \in \Omega$ such that $M(t) \ll_\delta 1$.

For any such t , E_t is 1-measurable with growth $\ll_\delta 1$. Indeed, note for any $r > 0$ that

$$|\{n \in [N] : |\phi(n) - t| \leq r\}| \ll_\delta rN.$$

Choosing $\eta : \mathbf{R} \rightarrow \mathbf{R}^+$ of Lipschitz norm $\|\eta\|_{\text{Lip}} \ll 1/r$ such that $\eta(x) = 0$ if $x < t - r$ and $\eta(x) = 1$ if $x > t + r$, it follows that $\|1_{E_t} - \eta \circ \phi\|_2 \ll_\delta \sqrt{r}$. Since ϕ is a function of θn of Lipschitz norm $\ll 1$, this implies that 1_{E_t} is 1-measurable with growth $\ll_\delta 1$. \square

A factor \mathcal{B} of $[N]$ is a subalgebra of $2^{[N]}$, or equivalently a partition of $[N]$ into cells. We say a factor \mathcal{B}' *refines* another \mathcal{B} if every cell of \mathcal{B} is a union of cells of \mathcal{B}' . We call \mathcal{B} a 1-*factor* with complexity at most M and growth \mathcal{F} if \mathcal{B} has M cells, each of which is 1-measurable with growth \mathcal{F} . Note in this case that every \mathcal{B} -measurable (i.e., constant on each cell of \mathcal{B}) function $f : [N] \rightarrow [-1, 1]$ is 1-measurable with growth $\ll_{M, \mathcal{F}} 1$.

For $x \in [N]$ we define $\mathcal{B}(x)$ to be the unique cell containing x , and we define the *conditional expectation* $\mathbf{E}(f|\mathcal{B})$ of a function $f : [N] \rightarrow \mathbf{C}$ by

$$\mathbf{E}(f|\mathcal{B})(x) = \frac{1}{|\mathcal{B}(x)|} \sum_{y \in \mathcal{B}(x)} f(y).$$

Equivalently, the function $\mathbf{E}(f|\mathcal{B})$ is the orthogonal projection of f onto the subspace of \mathcal{B} -measurable functions. Finally, with respect to a fixed function $f : [N] \rightarrow \mathbf{C}$, the *energy* of \mathcal{B} is $\mathcal{E}(\mathcal{B}) = \|\mathbf{E}(f|\mathcal{B})\|_2^2$.

Corollary 3 (Lack of uniformity allows energy increment). *Suppose \mathcal{B} is a 1-factor of complexity $\leq M$ and growth \mathcal{F} and $f : [N] \rightarrow [-1, 1]$ is a function such that $\|f - \mathbf{E}(f|\mathcal{B})\|_{U^2([N])} \geq \delta$. Then there exists a refinement \mathcal{B}' of \mathcal{B} of complexity $\leq 2M$ and growth $\ll_{M, \delta, \mathcal{F}} 1$ such that*

$$\mathcal{E}(\mathcal{B}') - \mathcal{E}(\mathcal{B}) \gg_\delta 1.$$

Proof. By the previous corollary there is a 1-measurable set $E \subset [N]$ with growth $\ll_\delta 1$ such that

$$|\langle f - \mathbf{E}(f|\mathcal{B}), 1_E \rangle| \gg_\delta 1.$$

Let \mathcal{B}' be the factor generated by \mathcal{B} and E . Then \mathcal{B}' is a 1-factor of complexity $\leq 2M$ and growth $\ll_{M, \delta, \mathcal{F}} 1$, and since 1_E is \mathcal{B}' -measurable we have

$$|\langle \mathbf{E}(f|\mathcal{B}') - \mathbf{E}(f|\mathcal{B}), 1_E \rangle| \gg_\delta 1.$$

Now Cauchy–Schwarz and the Pythagorean theorem imply that

$$\mathcal{E}(\mathcal{B}') - \mathcal{E}(\mathcal{B}) = \|\mathbf{E}(f|\mathcal{B}') - \mathbf{E}(f|\mathcal{B})\|_2^2 \gg_\delta 1. \quad \square$$

We can now deduce a weak form of the regularity lemma, occasionally referred to as the Koopman–von Neumann theorem.

Corollary 4 (Weak regularity). *Let \mathcal{B} be a 1-factor of complexity M and growth \mathcal{F} , and let $f : [N] \rightarrow [-1, 1]$ be a function. Then there exists a refinement \mathcal{B}' of \mathcal{B} of complexity $\ll_{\delta, M} 1$ and growth $\ll_{\delta, M, \mathcal{F}} 1$ such that*

$$\|f - \mathbf{E}(f|\mathcal{B}')\|_{U^2([N])} \leq \delta.$$

Proof. Repeatedly apply the previous corollary to refine the 1-factor \mathcal{B} . Since $0 \leq \mathcal{E}(\mathcal{B}) \leq 1$, this process must end after $\ll_\delta 1$ steps. \square

Finally, by iterating *this* result, we deduce full regularity.

Theorem 5 (The U^2 regularity lemma). *Let $f : [N] \rightarrow [0, 1]$ be a function, \mathcal{F} a growth function, and $\varepsilon > 0$. Then there is a quantity $M \ll_{\varepsilon, \mathcal{F}} 1$ and a decomposition*

$$f = f_{\text{str}} + f_{\text{sml}} + f_{\text{unf}}$$

of f into functions $f_{\text{str}}, f_{\text{sml}}, f_{\text{unf}} : [N] \rightarrow [-1, 1]$ such that

- (1) f_{str} has 1-complexity at most M ,
- (2) f_{sml} has $L^2([N])$ norm at most ε ,
- (3) f_{unf} has $U^2([N])$ norm at most $1/\mathcal{F}(M)$,
- (4) f_{str} and $f_{\text{str}} + f_{\text{sml}}$ take values in $[0, 1]$.

Proof. Starting with $M_0 = 1$ and $\mathcal{B}_0 = \{\emptyset, [N]\}$, suppose inductively that \mathcal{B}_i is a 1-factor of complexity and growth $\ll_{i, M_i, \mathcal{F}} 1$. Then there is a function $f_{\text{str}}^{(i)} : [N] \rightarrow \mathbf{R}$ of 1-complexity $M_{i+1} \ll_{\varepsilon, i, M_i, \mathcal{F}} 1$ such that $M_{i+1} \geq M_i$ and

$$\|\mathbf{E}(f|\mathcal{B}_i) - f_{\text{str}}^{(i)}\|_2 \leq \varepsilon/2.$$

Moreover, by truncating $f_{\text{str}}^{(i)}$ above and below (which doesn't increase 1-complexity) we may assume that $f_{\text{str}}^{(i)} : [N] \rightarrow [0, 1]$. By the previous corollary there is a refinement \mathcal{B}_{i+1} of \mathcal{B}_i of complexity and growth $\ll_{i, M_{i+1}, \mathcal{F}} 1$ such that

$$\|f - \mathbf{E}(f|\mathcal{B}_{i+1})\|_{U^2([N])} \leq 1/\mathcal{F}(M_{i+1}).$$

Note in the end that $M_i \ll_{\varepsilon, i, \mathcal{F}} 1$, and since $(\mathcal{E}(\mathcal{B}_i))$ is an increasing sequence in $[0, 1]$ there is some $i \ll_{\varepsilon} 1$ such that

$$\mathcal{E}(\mathcal{B}_{i+1}) - \mathcal{E}(\mathcal{B}_i) = \|\mathbf{E}(f|\mathcal{B}_{i+1}) - \mathbf{E}(f|\mathcal{B}_i)\|_2^2 \leq \varepsilon^2/4.$$

Let $M = M_{i+1}$ and let

$$\begin{aligned} f_{\text{str}} &= f_{\text{str}}^{(i)}, \\ f_{\text{sm1}} &= \mathbf{E}(f|\mathcal{B}_{i+1}) - f_{\text{str}}^{(i)}, \\ f_{\text{unf}} &= f - \mathbf{E}(f|\mathcal{B}_{i+1}). \end{aligned} \quad \square$$

It is often convenient to make the structure of f_{str} a little more explicit. Specifically, we know that $f_{\text{str}}(n) = F(\theta n)$ for some $F : \mathbf{T}^d \rightarrow [0, 1]$ and $\theta \in \mathbf{T}^d$ such that $d, \|F\|_{\text{Lip}} \leq M$, but what exactly this entails about the behaviour of f_{str} depends critically on the Diophantine properties of θ . For counting purposes we would like $\theta \in \mathbf{T}^d$ to be (A, N) -irrational for some large A , meaning that if $q \in \mathbf{Z}^d \setminus \{0\}$ and $\|q\|_1 \leq A$ (where if $q = (q_1, \dots, q_d)$ then $\|q\|_1 = |q_1| + \dots + |q_d|$) then $\|q \cdot \theta\|_{\mathbf{T}} \geq A/N$: this would guarantee that θn rapidly equidistributes over \mathbf{T}^d . Of course there are other possible behaviours of θ : it may be that θ itself is small, in which case $q \cdot \theta$ moves slowly away from 0, or it may be that θ is rational, in which case $q \cdot \theta$ frequently returns to 0, or there may be a combination of these behaviours. Nevertheless, it turns out that once these two pollutants are boiled off, the remnant is highly irrational in the above sense.

We say a subtorus T of \mathbf{T}^d of dimension d' has *complexity* at most M if there is some $L \in \text{SL}_d(\mathbf{Z})$, all of whose coefficients have size at most M , such that $L(T) = \mathbf{T}^{d'} \times \{0\}^{d-d'}$. In this case we implicitly identify T with $\mathbf{T}^{d'}$ using L . For instance, we say $\theta \in \mathbf{T}^d$ is (A, N) -irrational in T if $L(\theta)$ is (A, N) -irrational in $\mathbf{T}^{d'}$.

Theorem 6. *Given $\theta \in \mathbf{T}^d$, a positive integer N , and a growth function \mathcal{F} , there is a quantity $M \ll_{d, \mathcal{F}} 1$ and a decomposition*

$$\theta = \theta_{\text{smth}} + \theta_{\text{rat}} + \theta_{\text{irrat}}$$

such that

- (1) θ_{smth} is (M, N) -smooth, meaning $d(\theta_{\text{smth}}, 0) \leq \frac{M}{N}$,
- (2) θ_{rat} is M -rational, meaning $q\theta_{\text{rat}} = 0$ for some $q \leq M$, and
- (3) θ_{irrat} is $(\mathcal{F}(M), N)$ -irrational in a subtorus of complexity $\leq M$.

Proof. Starting with $M_0 = 1$, $\theta_{\text{smth}}^{(0)} = \theta_{\text{rat}}^{(0)} = 0$, $\theta_{\text{irrat}}^{(0)} = \theta$, and $T_0 = \mathbf{T}^d$, suppose inductively that

$$\theta = \theta_{\text{smth}}^{(i)} + \theta_{\text{rat}}^{(i)} + \theta_{\text{irrat}}^{(i)},$$

where $\theta_{\text{smth}}^{(i)}$ is (M_i, N) -smooth, $\theta_{\text{rat}}^{(i)}$ is M_i -rational, and $\theta_{\text{irrat}}^{(i)}$ lies in a subtorus T_i of dimension $d - i$ and complexity $\leq M_i$.

If $\theta_{\text{irrat}}^{(i)}$ is $(\mathcal{F}(M_i), N)$ -irrational in T_i then we are done, so suppose that $L \in \text{SL}_d(\mathbf{Z})$ is a linear map of complexity $\leq M_i$ identifying T with \mathbf{T}^{d-i} and such that

$$\|q \cdot L(\theta_{\text{irrat}}^{(i)})\|_{\mathbf{T}} \leq \frac{\mathcal{F}(M_i)}{N}$$

for some $q \in \mathbf{Z}^{d-i} \setminus \{0\}$ such that $\|q\|_1 \leq \mathcal{F}(M_i)$. Choose $\theta_{\text{smth}}^{(i)'} \in T$ so that

$$\|q \cdot L(\theta_{\text{irrat}}^{(i)} - \theta_{\text{smth}}^{(i)'})\|_{\mathbf{T}} = 0$$

and such that $d(L(\theta_{\text{smth}}^{(i)'}), 0) \leq \mathcal{F}(M_i)/N$, so $d(\theta_{\text{smth}}^{(i)'}, 0) \ll_{M_i, d, \mathcal{F}} 1/N$. Let $q = mq'$ where $m \in \mathbf{Z}^+$ and q' is primitive in \mathbf{Z}^{d-i} . Then

$$q' \cdot L(\theta_{\text{irrat}}^{(i)} - \theta_{\text{smth}}^{(i)'}) \in \frac{1}{m} \mathbf{Z}.$$

Now using the Euclidean algorithm, choose $\theta_{\text{rat}}^{(i)'} \in T$ so that

$$q' \cdot L(\theta_{\text{irrat}}^{(i)} - \theta_{\text{smth}}^{(i)'} - \theta_{\text{rat}}^{(i)'}) \in \mathbf{Z}$$

and such that $mL(\theta_{\text{rat}}^{(i)'}) = 0$, so that $m\theta_{\text{rat}}^{(i)'} = 0$. Finally, let

$$\begin{aligned} \theta_{\text{smth}}^{(i+1)} &= \theta_{\text{smth}}^{(i)} + \theta_{\text{smth}}^{(i)'}, \\ \theta_{\text{rat}}^{(i+1)} &= \theta_{\text{rat}}^{(i)} + \theta_{\text{rat}}^{(i)'}, \\ \theta_{\text{irrat}}^{(i+1)} &= \theta_{\text{irrat}}^{(i)} - \theta_{\text{smth}}^{(i)'} - \theta_{\text{rat}}^{(i)'}, \end{aligned}$$

and choose $M_{i+1} \ll_{M_i, d, \mathcal{F}} 1$ so that $\theta_{\text{smth}}^{(i+1)}$ is (M_{i+1}, N) -smooth, $\theta_{\text{rat}}^{(i+1)}$ is M_{i+1} -rational, and the subtorus $T_{i+1} = \{x \in T_i : q' \cdot L(x) = 0\}$ has complexity $\leq M_{i+1}$.

In the end note that $M_i \ll_{i, d, \mathcal{F}} 1$, and since T_i has dimension $d - i$ we can iterate this argument no more than d times, so for some $i \leq d$ we must have that $\theta_{\text{irrat}}^{(i)}$ is $(\mathcal{F}(M_i), N)$ -irrational in T_i . \square

We can now state and prove the irrational version of the regularity lemma. This version improves on Theorem 5 by giving f_{str} the structure

$$f_{\text{str}}(n) = F(n/N, n \bmod q, \theta n),$$

where

$$F : [0, 1] \times \mathbf{Z}/q\mathbf{Z} \times \mathbf{T}^d \rightarrow \mathbf{R},$$

$q, d, \|F\|_{\text{Lip}} \leq M$, and θ is $(\mathcal{F}(M), N)$ -irrational. Here we take the usual Euclidean metrics on $[0, 1]$ and \mathbf{T}^d , the discrete metric on $\mathbf{Z}/q\mathbf{Z}$, the sum of these metrics on $[0, 1] \times \mathbf{Z}/q\mathbf{Z} \times \mathbf{T}^d$, and then define $\|F\|_{\text{Lip}}$ as before.

Theorem 7 (U^2 regularity, irrational version). *Let $f : [N] \rightarrow [0, 1]$ be a function, \mathcal{F} a growth function, and $\varepsilon > 0$. Then there is a quantity $M \ll_{\varepsilon, \mathcal{F}} 1$ and a decomposition*

$$f = f_{\text{str}} + f_{\text{sml}} + f_{\text{unf}}$$

of f into functions $f_{\text{str}}, f_{\text{sml}}, f_{\text{unf}} : [N] \rightarrow [-1, 1]$ such that

- (1) $f_{\text{str}}(n) = F(n/N, n \bmod q, \theta n)$, where

$$F : [0, 1] \times \mathbf{Z}/q\mathbf{Z} \times \mathbf{T}^d \rightarrow [0, 1],$$

$q, d, \|F\|_{\text{Lip}} \leq M$, and $\theta \in \mathbf{T}^d$ is $(\mathcal{F}(M), N)$ -irrational,

- (2) f_{sml} has $L^2([N])$ norm at most ε ,

- (3) f_{unf} has $U^2([N])$ norm at most $1/\mathcal{F}(M)$,
- (4) f_{str} and $f_{\text{str}} + f_{\text{sml}}$ take values in $[0, 1]$.

Proof. Let \mathcal{F}_1 and \mathcal{F}_2 be growth functions depending on ε and \mathcal{F} in a manner to be determined. By Theorem 5 there exists $M_1 \ll_{\varepsilon, \mathcal{F}_1} 1$ and a decomposition

$$f = f_{\text{str}} + f_{\text{sml}} + f_{\text{unf}}$$

of f into functions $f_{\text{str}}, f_{\text{sml}}, f_{\text{unf}} : [N] \rightarrow [-1, 1]$ such that

- (1) $f_{\text{str}}(n) = F(\theta n)$, where $F : \mathbf{T}^d \rightarrow [0, 1]$, $d, \|F\|_{\text{Lip}} \leq M_1$, and $\theta \in \mathbf{T}^d$,
- (2) f_{sml} has $L^2([N])$ norm at most ε ,
- (3) f_{unf} has $U^2([N])$ norm at most $1/\mathcal{F}_1(M_1)$, and
- (4) f_{str} and $f_{\text{str}} + f_{\text{sml}}$ take values in $[0, 1]$.

Now by the previous theorem we can find $M_2 \ll_{M_1, \mathcal{F}_2} 1$ such that $M_2 \geq M_1$ and such that θ decomposes as

$$\theta = \theta_{\text{smth}} + \theta_{\text{rat}} + \theta_{\text{irrat}},$$

where

- (1) θ_{smth} is (M_2, N) -smooth, meaning $d(\theta_{\text{smth}}, 0) \leq \frac{M_2}{N}$,
- (2) θ_{rat} is M_2 -rational, meaning $q\theta_{\text{rat}} = 0$ for some $q \leq M_2$, and
- (3) θ_{irrat} is $(\mathcal{F}_2(M_2), N)$ -irrational in a subtorus of complexity $\leq M_2$.

Then

$$F(\theta n) = F(\theta_{\text{smth}}n + \theta_{\text{rat}}n + \theta_{\text{irrat}}n) = \tilde{F}(n/N, n \bmod q, nL(\theta_{\text{irrat}})),$$

where $\tilde{F} : [0, 1] \times \mathbf{Z}/q\mathbf{Z} \times \mathbf{T}^d \rightarrow [0, 1]$ is defined by

$$\tilde{F}(x, y, z) = F(N\theta_{\text{smth}}x + \theta_{\text{rat}}y + L^{-1}(z)).$$

Noting that $\|\tilde{F}\|_{\text{Lip}} \ll_{M_2} 1$, we can find $M \ll_{M_2} 1$ exceeding both M_2 and $\|\tilde{F}\|_{\text{Lip}}$. But since $M \ll_{M_2} 1$, if \mathcal{F}_2 is sufficiently large depending on \mathcal{F} then $\mathcal{F}_2(M_2) \geq \mathcal{F}(M)$, and similarly $M_2 \ll_{M_1, \mathcal{F}_2} 1$, so if \mathcal{F}_1 is sufficiently large depending on \mathcal{F}_2 then $\mathcal{F}_1(M_1) \geq \mathcal{F}_2(M_2) \geq \mathcal{F}(M)$. After all these dependencies are fixed we have $M \ll_{\varepsilon, \mathcal{F}} 1$, and the conclusion of the theorem holds. \square

In applications one typically combines the arithmetic regularity lemma with some sort of counting lemma such as the following. As mentioned already, if $\theta \in \mathbf{T}^d$ is highly irrational (i.e., (A, N) -irrational for large A), then the sequence θn is highly equidistributed over \mathbf{T}^d as n ranges over long progressions. This allows us to relate counts weighted by f_{str} to integrals of F .

Lemma 8. *Suppose that $\theta \in \mathbf{T}^d$ is (A, N) -irrational, and let $F : \mathbf{T}^d \rightarrow \mathbf{C}$ be a function with Lipschitz constant at most M . Suppose that $P \subset \{1, \dots, N\}$ is an arithmetic progression of length at least ηN . Then, provided that $A > A_0(M, d, \eta, \delta)$ is large enough,*

$$\left| \mathbf{E}_{n \in P} F(\theta n) - \int F d\mu \right| \leq \delta.$$

Proof. The key here (as usual in equidistribution theory) is to take a Fourier expansion of F and truncate it. In particular, we may find $M_0 = O_{M, d, \delta}(1)$ and coefficients c_m with $c_0 = \int F$ and $c_m = O_{M, d}(1)$ for $m \neq 0$ such that

$$\left| F(x) - \sum_{\|m\|_1 \leq M_0} c_m e(m \cdot x) \right| \leq \delta/2$$

uniformly in x . For a proof, see for example [GT08, Lemma A.9]. It follows, of course, that

$$\left| \mathbf{E}_{n \in P} F(\theta n) - \int F d\mu \right| \leq \sum_{\|m\|_1 \leq M_0, m \neq 0} |c_m| |\mathbf{E}_{n \in P} e(m \cdot \theta n)| + \frac{\delta}{2}.$$

Thus we need only show that

$$\mathbf{E}_{n \in P} e(m \cdot \theta n) = o_{m, \eta; A \rightarrow \infty}(1),$$

and then take A sufficiently large. If the common difference of the arithmetic progression P is h , then by summing the geometric progression we have the bound

$$\mathbf{E}_{n \in P} e(m \cdot \theta n) \ll \frac{1}{\eta N \|(m \cdot \theta)h\|_{\mathbf{T}}}.$$

But if $A > |h| \|m\|_1$ then, by the definition of (A, N) -irrationality, $\|(m \cdot \theta)h\|_{\mathbf{T}} \geq A/N$. Since $h \leq 2\eta^{-1}$, the result follows immediately. \square

If f_{str} has the structure given by Theorem 7 and \mathcal{F} grows sufficiently rapidly, then the triple $(n/N, n \bmod q, n\theta)$ is highly equidistributed over $[0, 1] \times \mathbf{Z}/q\mathbf{Z} \times \mathbf{T}^d$ as n ranges over $\{1, \dots, N\}$. Thus we have the following slightly more involved counting lemma, which is proved in essentially the same way.

Lemma 9. *Suppose that $\theta \in \mathbf{T}^d$ is (A, N) -irrational. Let $q \in \mathbf{N}$, and let $F : [0, 1] \times \mathbf{Z}/q\mathbf{Z} \times \mathbf{T}^d \rightarrow \mathbf{C}$ be a function with Lipschitz constant at most M . Let $\delta > 0$ be arbitrary. Then, provided that $A > A_0(M, q, d, \delta)$ and $N > N_0(M, q, d, \delta)$ are large enough,*

$$\left| \mathbf{E}_{n \leq N} F(n/N, n \bmod q, \theta n) - \int F d\mu \right| \leq \delta.$$

Proof sketch. Again the idea is to take a truncated Fourier expansion of F , but because $F|_{\{0\} \times \mathbf{Z}/q\mathbf{Z} \times \mathbf{T}^d}$ and $F|_{\{1\} \times \mathbf{Z}/q\mathbf{Z} \times \mathbf{T}^d}$ need not agree the expansion looks a little more complicated. However, F can be extended to an M -Lipschitz function $[-1, 1] \times \mathbf{Z}/q\mathbf{Z} \times \mathbf{T}^d \rightarrow \mathbf{C}$ such that $F(x, y, z) = F(-x, y, z)$, so F may be approximated by a sum of the functions $\phi_{k,a,m}$ given by

$$\phi_{k,a,m}(x, y, z) = e\left(\frac{m}{2}x + \frac{a}{q}y + m \cdot z\right) + e\left(-\frac{m}{2}x + \frac{a}{q}y + m \cdot z\right), \quad (1)$$

where $k \in \mathbf{Z}$, $a \in \mathbf{Z}/q\mathbf{Z}$, $m \in \mathbf{Z}^d$. Then just as in the proof of the previous lemma we need only check that

$$\mathbf{E}_{n \leq N} \phi_{k,a,m}(n/N, n \bmod q, \theta n) = o_{k,a,m,q;A,N \rightarrow \infty}(1) \quad (2)$$

provided that k, a, m are not all zero. Substituting in, the left-hand side is

$$\mathbf{E}_{n \leq N} \left(e\left(\left(\frac{k}{2N} + \frac{a}{q} + m \cdot \theta\right)n\right) + e\left(\left(-\frac{k}{2N} + \frac{a}{q} + m \cdot \theta\right)n\right) \right). \quad (3)$$

Summing the geometric progressions, we see that this is bounded by ε unless

$$\left\| \pm \frac{k}{2N} + \frac{a}{q} + m \cdot \theta \right\|_{\mathbf{T}} \ll \frac{1}{N\varepsilon} \quad (4)$$

for either choice of sign.

Supposing first that $m \neq 0$, inequality (4) implies

$$\left\| \pm \frac{mk}{2N} + qm \cdot \theta \right\|_{\mathbf{T}} \ll \frac{q}{N\varepsilon},$$

and hence

$$\|m' \cdot \theta\|_{\mathbf{T}} \ll \frac{q}{N\varepsilon} + \frac{mq}{2N},$$

where $m' = qm$. But if A is sufficiently large in terms of ε, q, k, m , this is contrary to the (A, N) -irrationality of θ .

Hence suppose that $m = 0$. Then if N is large enough depending on m and q , (4) implies that $a = 0$. Thus $a = m = 0$, so $k \neq 0$. But then the expression (3) is

$$\mathbf{E}_{n \leq N} (e(kn/2N) + e(-kn/2N)) = O_k(1/N),$$

so (2) certainly follows in this case as well. \square

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